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CONVERGENCE OF DOUBLE FOURIER SERIES AND W-CLASSES

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Abstract. The double Fourier series of functions of the generalized bounded variation class $\{n/\ln(n+1)\}^*BV$ are shown to be Pringsheim convergent everywhere. In a certain sense, this result cannot be improved. In general, functions of class Λ^*BV, defined here, have quadrant limits at every point and, for $f \in \Lambda^*BV$, there exist at most countable sets P and Q such that, for $x \notin P$ and $y \notin Q$, f is continuous at (x, y). It is shown that the previously studied class ΛBV contains essentially discontinuous functions unless the sequence Λ satisfies a strong condition.

1. Introduction

A remarkable variety of definitions of bounded variation have been given for functions of two variables. Here we will discuss generalizations of these definitions along the lines of the notion of Λ-bounded variation (ΛBV) in one variable introduced by Waterman. He used it to extend the Dirichlet-Jordan theorem, and we will investigate the analogous problem for double Fourier series.

For an excellent discussion of ΛBV and its relation to other generalizations of bounded variation, see Avdispahić [1]. For applications to summability and Tauberian theorems, see [2, 8, 11].

Definition 1. Let $\Lambda = \{\lambda_k\}_1^\infty$ be a monotone nondecreasing sequence of positive numbers such that

$$\sum_1^\infty \lambda_k^{-1} = \infty,$$

and let Y denote the class of such sequences. A real function f defined on an interval $[a, b]$ is said to be of Λ-bounded variation, $f \in \Lambda BV([a, b])$, if

$$V_\Lambda(f; [a, b]) = \sup_{\mathcal{I},n} \sum_1^n \frac{|f(I_k)|}{\lambda_k} = \sup_{\mathcal{I},n} \sum_1^n \frac{|f(\beta_k) - f(\alpha_k)|}{\lambda_k} < \infty,$$

where $\mathcal{I}$ denotes the class of collections of nonoverlapping intervals $\{I_k = [\alpha_k, \beta_k] \subset [a, b],\ k = 1, \dots, n\}$.

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Note that functions of Λ-bounded variation are bounded, have right and left limits at every point, and so their discontinuities are at most countable [9].

Function classes, whose definitions depend on the boundedness of sums of the absolute values of interval functions multiplied by weights from sequences such as Λ, have come to be known as *W-classes.*

In [7, 10], Waterman proved the following generalization of the Dirichlet-Jordan theorem.

Theorem A. *If f is a $2\pi-$periodic function, $H = \{n\}_1^\infty$, $T = [-\pi, \pi]$ and $f \in HBV(T)$, then $S[f]$, the Fourier series of f, converges at every point and, if I is a closed interval of points of continuity, then $S[f]$ converges uniformly on I. If $\Lambda BV \setminus HBV \neq \emptyset$, then there is an $f \in \Lambda BV(T)$ such that $S[f]$ diverges at a point.*

A definition of ΛBV for two variables which has been used by Saakjan [5] and Sablin [6] is as follows.

Definition 2. Let $\Lambda \in Y$ and let f be a measurable function on the rectangle $A = [a, b] \times [c, d]$. Then $f \in \Lambda BV(A)$ if and only if

(1) $f(\cdot, c) \in \Lambda BV([a, b])$ and $f(a, \cdot) \in \Lambda BV([c, d])$, and
(2) if $\mathcal{I}_1$ and $\mathcal{I}_2$ are the sets of finite collections of nonoverlapping intervals $I_k = [\alpha_k, \beta_k]$ and $I_j = [\gamma_j, \delta_j]$ in $[a, b]$ and $[c, d]$ respectively and $f(I_k \times I_j) = f(\alpha_k, \gamma_j) - f(\alpha_k, \delta_j) - f(\beta_k, \gamma_j) + f(\beta_k, \delta_j)$, then

$$V_\Lambda(f; [a, b]) = \sup_{\mathcal{I}_1, \mathcal{I}_2} \sum_k \sum_j \frac{|f(I_k \times I_j)|}{\lambda_k \lambda_j} < \infty.$$

Remark 1. If $\lambda_k \equiv 1$, or what is the same, $\lambda_k = O(1)$, $\Lambda BV(A)$ is the set of functions of Hardy-Krause bounded variation on A.

It is clear that the functions of $\Lambda BV(A)$ are bounded, but the question of continuity is more complicated than in the case of functions of one variable. Dyachenko [3] has proved the following theorem.

Theorem B. *The following conditions are equivalent:*

(i) *for any $f \in \Lambda BV(T^2)$ there exist two at most countable subsets A and B of T such that f is continuous at every point $(x, y) \in T^2$ such that $x \notin A$ and $y \notin B$;*

(ii) *for any $f \in \Lambda BV(T^2)$ and any $(x_0, y_0) \in T^2$, $\lim f(x, y)$ exists as $(x, y) \to (x_0, y_0)$ in each of the open coordinate quadrants;*

(iii) $\sum_1^\infty \lambda_k^{-2} = \infty$.

(The third condition will be called **Condition** $(*)$.)

Thus we see, for example, that the characteristic function of the triangle $B = \{(x, y) \in [0, 1]^2, 0 \leq y \leq 1 - x\}$ is in $\Lambda BV([0, 1]^2)$ only if Condition $(*)$ does not hold.

If Λ does not satisfy (iii), the requirement of measurability cannot be omitted from Definition 2, for if it were, $\Lambda BV(A)$ would include functions not Lebesgue measurable. Even under the assumption of measurability, we show in Section 2 that $\Lambda BV(A)$ contains an everywhere discontinuous function and, moreover, a function f such that if $g = f$ a.e., then g is a.e. discontinuous.

We will consider the Pringsheim convergence of double Fourier series. If $f \in L(T^2)$ is 2π-periodic in each variable, then

$$S[f] = \sum_{m,n} a_{mn} e^{i(mx+ny)}$$

is its Fourier series, where

$$a_{mn} = a_{mn}(f) = \frac{1}{(2\pi)^2} \int_{T^2} f(x,y) e^{-i(mx+ny)} dxdy.$$

The rectangular partial sums of this series are

$$S_{N_1 N_2}(f;x,y) = \sum_{|k_1| \le N_1} \sum_{|k_2| \le N_2} a_{k_1 k_2} e^{i(k_1 x + k_2 y)}$$

with $N_1, N_2 \ge 0$. If $N_1 = N_2$, these are called *square sums.* If

$$S_{N_1 N_2}(f;x,y) \to \alpha \text{ as } \min(N_1, N_2) \to \infty,$$

we say that the Fourier series of f *converges to* α *at* (x,y) *in the Pringsheim sense.*

A.A. Saakyan [5] has shown

Theorem C. *If* $f \in HBV(T^2)$, *then the rectangular partial sums of* $S[f]$ *are uniformly bounded, and, if for* $(x_o, y_0) \in T^2$ *the limits of* $f(x,y)$*exist as* $(x,y) \to (x_o, y_0)$ *in each of the open coordinate quadrants, then* $S[f]$ *converges (Pringsheim) to the arithmetic mean of these limits.*

This result has been generalized to higher dimensions by A.I. Sablin [6].

As we shall see in §2, an $f \in HBV(T^2)$ need not have a point of continuity. For such a function, Theorem C is inapplicable.

We shall define another W-class such that functions of this class are continuous a.e., and prove that a theorem analogous to Theorem C holds for this class.

Definition 3. Let $\Lambda \in Y$ and let f be a real function on $A = [a,b] \times [c,d]$. We say $f \in \Lambda^* BV(A)$ if

(i) $f(\cdot, c) \in \Lambda BV([a,b])$ and $f(a, \cdot) \in \Lambda BV([c,d])$

and, if Γ is the set of finite collections of nonoverlapping rectangles $A_k = [\alpha_k, \beta_k] \times [\gamma_k, \delta_k] \subset A$ and $f(A_k) = f(\alpha_k, \gamma_k) - f(\alpha_k, \delta_k) - f(\beta_k, \gamma_k) + f(\beta_k, \delta_k)$, then

(ii) $V_\Lambda^*(f;A) = \sup_\Gamma \sum_k \frac{|f(A_k)|}{\lambda_k} < \infty$.

For $f \in \Lambda^* BV(A)$ we set

(iii) $\|f\|_{\Lambda^*} = \|f\|_{\Lambda^*(A)} = |f(a,c)| + V_\Lambda(f(\cdot, c)) + V_\Lambda(f(a, \cdot)) + V_\Lambda^*(f;A)$.

Remark 2. Note that if $f \in \Lambda^* BV(A)$, then

$$V_\Lambda(f(\cdot, y); [a,b]) \le V_\Lambda^*(f;A) + V_\Lambda(f(\cdot, c); [a,b])$$

for every $y \in [c,d]$. The analogous result holds for the Λ-variation of the restriction of f to the vertical segments.

In §3 we shall prove

Theorem 1. *Let* $\Lambda \in Y$ *and* $A = [a,b] \times [c,d]$. *Then, for any* $f \in \Lambda^* BV(A)$,

(i) *there exist at most countable sets* $P \subset [a,b]$ *and* $Q \subset [c,d]$ *such that* f *is continuous at every* $(x,y) \in A$ *such that* $x \notin P$ *and* $y \notin Q$; *and*

(ii) *at every point* $(x_0, y_0) \in A$, $\lim_{(x,y) \to (x_0, y_0)} f(x,y)$ *exists in each open coordinate quadrant.*

In that section we also discuss the relation between $\Lambda^*BV(A)$ and $\Lambda BV(A)$.

In §4 we study the convergence of Fourier series of functions of class $\Lambda^*BV(A)$, and prove

Theorem 2. *Let f be a real function on $\mathbb{R}^2$ which is 2π-periodic in each variable and is in $\Lambda^*BV(T^2)$ with $\Lambda = \{\frac{n}{\ln(n+1)}\}$. Then the rectangular partial sums of $S[f]$ are uniformly bounded and converge at each point to the arithmetic mean of the quadrant limits.*

We also show that, in a certain sense, this result cannot be improved.

Theorem 3. *Let $\Lambda = \left\{\frac{n}{\ln(n+1)}\xi_n\right\} \in Y$, where $\xi_n \uparrow \infty$ as $n \to \infty$. Then there exists a function $f \in \Lambda^*BV(T^2)$ such that the square partial sums of its Fourier series diverge unboundedly at $(0,0)$.*

2. Discontinuous functions in W-classes

We shall require the following lemmas.

Lemma 1. *Let $\Lambda \in Y$ be such that Condition $(*)$ does not hold (i.e., $\sum \lambda_k^{-2} < \infty$) and let $A = [a,b]\times[c,d]$ be a nondegenerate interval. Suppose $E \subset A$ has a connected intersection with every horizontal and vertical line. Then χ_E, the characteristic function of E, is in $\Lambda BV(A)$ and $V_\Lambda(\chi_E; A) < C < \infty$, where C is an absolute constant.*

Proof. Let $\{I_k\}_1^n = \{[\alpha_k, \beta_k]\}$ in $[a,b]$ and $\{J_r\}_1^m = \{[\gamma_r, \delta_r]\}$ in $[c,d]$ be two collections of nonoverlapping intervals. Then, for each $k = 1, 2, \dots, n$, in the sum

$$S = \sum_{k=1}^{n}\sum_{r=1}^{m} |\chi_E(I_k \times J_r)| / \lambda_k \lambda_r,$$

there are at most four different $r_{k,j}$ for which $\left|\chi_E(I_k \times J_{r_{k,j}})\right| \neq 0$, and in these cases it is either 1 or 2. Let

$$S_j = \sum_{k=1}^{n} \frac{\left|\chi_E(I_k \times J_{r_{k,j}})\right|}{\lambda_k \lambda_{r_{k,j}}}, \quad \text{for } j = 1,2,3,4.$$

Then $S = S_1 + S_2 + S_3 + S_4$, and, as each j can be associated with at most four $r_{k,j}$, we have

$$S_j \le 2\sum_{k=1}^{n} \frac{1}{\lambda_k \lambda_{r_{k,j}}} \le \sum_{k=1}^{n} \frac{1}{\lambda_k^2} + \sum_{k=1}^{n} \frac{1}{\lambda_{r_{k,j}}^2} \le 5\sum_{k=1}^{n} \frac{1}{\lambda_k^2} < C < \infty,$$

and, since the one-dimensional Λ-variation of χ_E on the edges of A is at most $2/\lambda_1$, Lemma 1 is established. □

Lemma 2. *$A = [a,b] \times [c,d]$ be a nondegenerate interval. There is a sequence of closed rectangles $\{A_i = I_i \times J_i\}$ in A with $I_i \cap I_j = \emptyset$ and $J_i \cap J_j = \emptyset$ for $i \neq j$, with*

$$\sum_{i=1}^{\infty} |I_i| < (b-a)/4 \quad \textit{and} \quad \sum_{i=1}^{\infty} |J_i| < (d-c)/4,$$

such that every neighborhood of each point of the following contains some A_i:

$$B = A \setminus \bigcup_{i=1}^{\infty}((I_i \times [c,d]) \cup ([a,b] \times J_i)).$$

Proof. Choose $n_1 = 1$ and consider the rectangles

$$E_{k,r,1} = \left[a + \frac{(b-a)(k-1)}{2^{n_1}}, a + \frac{(b-a)k}{2^{n_1}}\right] \times \left[c + \frac{(d-c)(r-1)}{2^{n_1}}, c + \frac{(d-c)r}{2^{n_1}}\right],$$

where $k, r = 1, 2 \ldots, 2^{n_1}$.

Choose an integer $n_2 > 2n_1 + 3$ and, for each choice of k and $r, 1 \le k, r \le 2^{n_1}$, choose a rectangle

$$\begin{aligned} &E'_{k,r,1} = [a_{k,r,1}, b_{k,r,1}] \times [c_{k,r,1}, d_{k,r,1}] \\ = &\left[a + \frac{(b-a)(l_k-1)}{2^{n_2}}, a + \frac{(b-a)l_k}{2^{n_2}}\right] \times \left[c + \frac{(d-c)(s_r-1)}{2^{n_2}}, c + \frac{(d-c)s_r}{2^{n_2}}\right] \subset E_{k,r,1} \end{aligned}$$

so that the projections of the chosen rectangles on the coordinate axes do not touch. Clearly

$$\sum_{k,r=1}^{2^{n_1}} (b_{k,r,1} - a_{k,r,1}) < \frac{b-a}{8} \text{ and } \sum_{k,r=1}^{2^{n_1}} (d_{k,r,1} - c_{k,r,1}) < \frac{d-c}{8}.$$

Next we consider the rectangles

$$E_{k,r,2} = \left[a + \frac{(b-a)(k-1)}{2^{n_2}}, a + \frac{(b-a)k}{2^{n_2}}\right] \times \left[c + \frac{(d-c)(r-1)}{2^{n_2}}, c + \frac{(d-c)r}{2^{n_2}}\right],$$

where k and r are chosen, $1 \le k, r \le 2^{n_2}$, so that

$$E_{k,r,2} \cap \bigcup_{k,r=1}^{2^{n_1}} ([a_{k,r,1}, b_{k,r,1}] \times [c, d]) \cup ([a, b] \times [c_{k,r,1}, d_{k,r,1}]) = \emptyset.$$

We then choose an integer $n_3 > 2n_2 + 4$ and, for each k and $r, 1 \le k, r \le 2^{n_2}$, just chosen, we choose a rectangle

$$\begin{aligned} &E'_{k,r,2} = [a_{k,r,2}, b_{k,r,2}] \times [c_{k,r,2}, d_{k,r,2}] \\ = &\left[a + \frac{(b-a)(l_k-1)}{2^{n_3}}, a + \frac{(b-a)l_k}{2^{n_3}}\right] \times \left[c + \frac{(d-c)(s_r-1)}{2^{n_3}}, c + \frac{(d-c)s_r}{2^{n_3}}\right] \\ &\subset E_{k,r,2} \end{aligned}$$

so that the projections of the chosen rectangles on the coordinate axes do not touch. Then

$$\sum_{k,r} (b_{k,r,2} - a_{k,r,2}) < \frac{b-a}{16} \text{ and } \sum_{k,r} (d_{k,r,2} - c_{k,r,2}) < \frac{d-c}{16}.$$

Proceeding inductively, we can define $E'_{k,r,,j}$, $j = 1, 2, \ldots$, choosing $n_{j+1} > 2n_j + j + 2$ at each step, and, renumbering the $E'_{k,r,,j}$ as we wish, $\{A_i\} = \{E'_{k,r,,j}\}$ is the required sequence of intervals. □

We turn now to the principal results of this section.

Proposition 1. *Suppose $\Lambda \in Y$, $A = [a, b] \times [c, d]$ is a nondegenerate rectangle and Condition $(*)$ does not hold. Then there exists an $f \in \Lambda BV(A)$ which is everywhere discontinuous.*

Proof. Let P and Q be the sets of rationals in $[a,b]$ and $[c,d]$ respectively. Divide the rectangle A into four quarters by passing lines parallel to the axes through the midpoints of the sides. In each of the rectangles A_i, $i=1,2,3,4$, thus formed, we select $(p_i,q_i), p_i \in P, q_i \in Q$, so that the $\{p_i\}$ and $\{q_i\}$ are distinct. Now we can quarter each A_i and choose one point in each sixteenth not yet containing a chosen point to form $\{(p_i,q_i)\}_5^{16}$, $p_i \in P, q_i \in Q$, so that $\{p_i\}_1^{16}$ and $\{q_i\}_1^{16}$ are sets of distinct points. Proceeding inductively, we obtain a dense set of points $E=\{(p_i,q_i)\}$ such that $p_i \neq p_j$ and $q_i \neq q_j$ when $i \neq j$. Thus any line parallel to an axis meets E in at most one point and, by Lemma 1, $\chi_E \in \Lambda BV(A)$. $\square$

The function we have constructed in Proposition 1 is everywhere discontinuous but is equivalent to the function identically equal to zero. The next result shows that in a class $\Lambda BV(A)$ in which Condition $(*)$ does not hold there exist *essentially discontinuous* functions.

Proposition 2. *Suppose* $\Lambda \in Y$, $A=[a,b]\times[c,d]$ *is a nondegenerate rectangle and Condition* $(*)$ *does not hold. Then there is an* $f \in \Lambda BV(A)$ *such that* $g=f$ *a.e. implies that* g *is a.e. discontinuous.*

Proof. Apply Lemma 2 to form the sequence $\{A_i\}$ and the set $B_1=B$. If $F_1=\bigcup\limits_1^\infty A_i$ and $f_1=\chi_{F_1}$, we see that $V_\Lambda(f_1)=C<\infty$, and we observe that $|A\setminus B_1|<|A|/2$. The set $A\setminus B_1$ can be divided into rectangles $\{D_i\}_{i=1}^\infty$ in the natural way. By applying Lemma 2 to each of the rectangles D_i, we obtain for each i a sequence $\{A_{ij}\}_{j=1}^\infty \subset D_i$ with similar properties. Let $F_{2,i}=\bigcup\limits_{j=1}^\infty A_{i,j}$ and $f_{2,i}=\chi_{F_{2,i}}, i=1,2,\dots$. Note that $V_\Lambda(f_{2,i})\le C$ for every i. If $A_{i,j}=[a_{i,j},b_{i,j}]\times[c_{i,j},d_{i,j}]$, set

$$B_{2,i}=D_i\diagdown\bigcup_{j=1}^\infty(([a_{i,j},b_{i,j}]\times[c,d])\cup([a,b]\times[c_{i,j},d_{i,j}])),\quad i=1,2,\dots\ .$$

Let us replace i by the symbol i_1. At the third stage we obtain sets $B_{3,i_1,,i_2}$, and as the induction proceeds we obtain sets $B_{r,i_1,\dots i_{r-1}}$. Let

$$U=B_1\cup(\bigcup_{r=2}^\infty(\bigcup_{i_1,\dots i_{r-1}=1}^\infty B_{r,i_1,\dots i_{r-1}}));$$

then $|A\diagdown U|=0$. We continue inductively to obtain functions $f_1,\{f_{2,i_1}\}_{i_1=1}^\infty,\dots,$ $\{f_{r,i_1,\cdots i_{r-1}}\}_{i_1,\cdots i_{r-1}=1}^\infty,\dots$, and we renumber these functions to form $\{h_k\}_{k=1}^\infty$. Now, letting

$$f=\sum_{k=1}^\infty 3^{-k}h_k\ ,$$

we have $f\in\Lambda BV(A)$.

Let $g=f$ a.e. and let E be the set of points in A at which g and f are equal. Clearly E is dense in A. We write $\omega(f;(x,y);E)$ for the oscillation of f at (x,y) over the set E and $\omega(g;(x,y);A)$ for the oscillation of g at (x,y) over the set A. Note that

$$\omega(g;(x,y);A)\ge\omega(f;(x,y);E).$$

Consider a point $(x,y)\in U$. Then (x,y) is in some $B_{r,i_1,\dots i_{r-1}}$. If k is such that $h_k=f_{r,i_1,\dots i_{r-1}}$, then $\omega(h_k;(x,y);E)=1$. Note that, since $h_l(A)=\{0,1\}$, for every

$(s,t) \in A$ and every l, we have either

$$\omega(h_l;(s,t);E) = 0 \text{ or } \omega(h_l;(s,t);E) = 1.$$

Let

$$k_0 = k_0(x,y) = \min\{l : \omega(h_l;(x,y);E) = 1\}.$$

Then

$$\omega(f;(x,y);E) \geq 3^{-k_0} - \sum_{l=k_0+1}^{\infty} 3^{-l}\omega(h_l;(x,y);E) \geq 3^{-k_0} - \sum_{l=k_0+1}^{\infty} 3^{-l} \geq 3^{-k_0-1},$$

implying that g is discontinuous at (x,y). □

3. Continuity properties of functions of $\Lambda^* BV$

We now turn our attention to the proof of Theorem 1.

Proof. Suppose there are infinitely many points (x_i,y_i) such that $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$, at which the oscillation of f exceeds $1/k$, k a natural number. For a natural number N we can find points (α_i,β_i) and (γ_i,δ_i), $i = 1,2,\ldots,N$, such that $f(\alpha_i,\beta_i) - f(\gamma_i,\delta_i) > 1/k$ and the sequences $\alpha_1,\gamma_1,\alpha_2,\gamma_2,\alpha_3,\ldots$ and $\beta_1,\delta_1,\beta_2,\delta_2,\beta_3,\ldots$ are strictly monotone. We will assume them to be increasing; the other cases are handled similarly.

We have

$$\begin{aligned} S_1 &= \sum_{i=0}^{N} \frac{|f(\alpha_i,\delta_i) - f(\gamma_i,\delta_i)|}{\lambda_i} \\ &\leq \sum_{i=0}^{N} \frac{|f((\alpha_i,\gamma_i)\times(c,\delta_i))|}{\lambda_i} + \sum_{i=0}^{N} \frac{|f(\alpha_i,c) - f(\gamma_i,c)|}{\lambda_i} \leq \|f\|_{\Lambda^*} \end{aligned}$$

and, in a similar fashion,

$$S_2 = \sum_{i=0}^{N} \frac{|f((\gamma_i,\beta_i) - f(\gamma_i,\delta_i))|}{\lambda_i} \leq \|f\|^*_{\Lambda^*} \ .$$

Thus

$$\begin{aligned} V_{\Lambda^*}(f;A) &\geq \sum_{i=0}^{N} \frac{|f((\alpha_i,\gamma_i)\times(\beta_i,\delta_i))|}{\lambda_i} \\ &\geq \sum_{i=0}^{N} \frac{|f(\alpha_i,\beta_i) - f(\gamma_i,\delta_i)|}{\lambda_i} - S_1 - S_2 \\ &\geq \frac{1}{k}\sum_{i=0}^{N}\frac{1}{\lambda_i} - 2\,\|f\|_{\Lambda^*}\ , \end{aligned}$$

which is false for N sufficiently large. Thus all points at which f has an oscillation greater than $1/k$ lie on a finite number of lines parallel to the axes, which establishes the first part of the theorem.

To establish the second part of Theorem 1, we assume that there is a point $p \in A$ such that $f(x,y)$ does not have a limit as $(x,y) \to p$ within an open coordinate quadrant with vertex p. Without loss of generality we may assume that $p = (0,0)$ and the quadrant is $\{(x,y) : x > 0, y > 0\}$.

Then there is an $\varepsilon > 0$ such that, for every $\delta > 0$, in every square $(0,\delta)^2$ the oscillation of f is greater than ε. Choose $s,t > 0$. Then, since f is in ΛBV in each variable separately, $\lim_{y\downarrow 0} f(s,y)$ and $\lim_{x\downarrow 0} f(x,t)$ exist. Choose $\delta > 0$ so that the oscillations of $f(s,y)$ and $f(x,t)$ on $0 < y < \delta$ and $0 < x < \delta$ respectively are less than $\varepsilon/8$. Now choose points (x_1,y_1) and (x_2,y_2) in $(0,\delta)^2$ so that

$$|f(x_1,y_1) - f(x_2,y_2)| > \varepsilon/2.$$

Then, letting $P = f(s,t) - f(s,y_1) - f(x_1,t) + f(x_1,y_1)$ and $Q = f(s,t) - f(s,y_2) - f(x_2,t) + f(x_2,y_2)$, we have

$$\begin{aligned} |P-Q| &\geq |f(x_1,y_1) - f(x_2,y_2)| - |f(x_1,t) - f(x_2,t)| - |f(s,y_1) - f(s,y_2)| \\ &> \varepsilon/2 - \varepsilon/8 - \varepsilon/8 = \varepsilon/4, \end{aligned}$$

so that at least one of $|P|$ and $|Q|$ exceeds $\varepsilon/8$, and so we have obtained a rectangle $A_1 \in (0,\delta)^2$ for which $|f(A_1)| > \varepsilon/8$. By choosing our points $(s,t), (x_1,y_1)$ and (x_2,y_2) sufficiently close to the origin, we can repeat this process to obtain a rectangle A_2 which does not overlap A_1 for which $|f(A_2)| > \varepsilon/8$. Thus we can form a sequence $\{A_n\}$of nonoverlapping intervals in $(0,\delta)^2$ with $|f(A_n)| > \varepsilon/8$. Then

$$\sum_{i=0}^{N} \frac{|f(A_k)|}{\lambda_k} > \frac{\varepsilon}{8}\sum_{i=0}^{N}\frac{1}{\lambda_k} \to \infty \quad \text{as } N \to \infty,$$

contradicting our assumption that $f \in \Lambda^* BV(A)$, and completing the proof of Theorem 1. □

It is natural to ask how the classes ΛBV and $\Lambda^* BV$ are related. This is by no means obvious, although they are clearly the same if $\{\lambda_i\}$ is bounded. There is no loss of generality in assuming the rectangle A to be $[0,1]^2$.

Proposition 3. *If $\Lambda \in Y$ is an unbounded sequence, then $\Lambda BV \setminus \Lambda^* BV \neq \emptyset$.*

Proof. First we consider the case where $\sum\limits_1^\infty \lambda_i^{-2} < \infty$ and consider χ_E, where

$$E = \{(x,y) \in [0,1]^2, y \leq 1-x\}.$$

Lemma 1 implies that $\chi_E \in \Lambda BV$, but from Theorem 1 we have $\chi_E \notin \Lambda^* BV$.

Now assume that Condition $(*)$ holds and $\lambda_i \to \infty$ as $i \to \infty$. We choose $\alpha_i \searrow 0$ so that

$$\sum_i \frac{\alpha_i}{\lambda_i} = \infty \quad \text{and} \quad \sum_i \frac{\alpha_i}{\lambda_i^2} < \infty.$$

Let

$$f = \sum_1^\infty \alpha_n \chi_{E_n}, \quad \text{where } E_n = [\frac{1}{2n}, \frac{1}{2n-1}]^2.$$

The rectangles $A_n = [2/(4n+1), 1/2n]^2$ are pairwise disjoint and $|f(A_n)| = \alpha_n$. Hence

$$\sum_{i=1}^{N} \frac{|f(A_i)|}{\lambda_i} = \sum_{i=1}^{N} \frac{\alpha_i}{\lambda_i} \to \infty \quad \text{as } N \to \infty,$$

implying $f \notin \Lambda^* BV$.

Clearly, $f(\cdot,0)$ and $f(0,\cdot)$ are in $\Lambda BV([0.1])$. Suppose $\{I_i\}_1^N$ and $\{J_i\}_1^N$ are collections of nonoverlapping intervals in $[0,1]$. For each I_i, there are no more than four values of j such that $f(I_i \times J_j) \neq 0$. Let $j(i)$ denote the smallest. Let $k(i)$

be the smallest of the indices of E_n such that $\chi_{E_n}(I_i \times J_j) \neq 0$ for some j. Then $|f(I_i \times J_j)| \leq 2\alpha_{k(i)}$. Also, each k can appear no more than twice as a $k(i)$ and each j can appear no more than twice as a $j(i)$. Thus

$$S = \sum_{1,j=1}^{N} \frac{|f(I_i \times J_j)|}{\lambda_i \lambda_j} \leq 8 \sum_{i=1}^{N} \frac{\alpha_{k(i)}}{\lambda_i \lambda_{j(i)}} \leq 8 \left(\sum_{i=1}^{N} \frac{\alpha_{k(i)}}{\lambda_i^2} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} \frac{\alpha_{k(i)}}{\lambda_{j(i)}^2} \right)^{\frac{1}{2}} \leq 32 \sum_{i=1}^{N} \frac{\alpha_i}{\lambda_i^2},$$

which is bounded above independently of N and the choice of $\{I_i\}$ and $\{J_j\}$. Thus $f \in \Lambda BV$. □

Proposition 4. *If $A = [a,b] \times [c,d]$, $0 < \alpha \leq 1$, and $\Lambda_\alpha = \{n^\alpha\}_{n=1}^\infty$, then*

$$\Lambda_\alpha^* BV \setminus \Lambda_\alpha BV \neq \emptyset.$$

Proof. We will assume once again the $A = [0,1]^2$. Let

$$f_n = \sum_{k=1}^{n} \sum_{1 \leq l \leq n/k} (-1)^{k+l} \chi_{[\frac{k-1}{n}, \frac{k}{n}) \times [\frac{l-1}{n}, \frac{l}{n})}.$$

C will denote a constant, not necessarily the same at each occurrence, and C_α a constant depending on α. The number of terms in the sum defining f_n is not greater than $n \ln(n+1)$, so for $\alpha \in (0,1)$,

$$V_{\Lambda_\alpha^*}(f_n) \leq C \sum_{r=1}^{n \ln(n+1)} r^{-\alpha} \leq C_\alpha (n \ln(n+1))^{1-\alpha}.$$

Similarly,

$$V_{\Lambda_1^*}(f_n) \leq C \ln(n+1).$$

On the other hand, for $\alpha \in (0,1)$,

$$V_{\Lambda_\alpha}(f_n) \geq C \sum_{k=1}^{n} k^{-\alpha} \left(\sum_{j=1}^{n/k} j^{-\alpha} \right) \geq C_\alpha \sum_{k=1}^{n} k^{-\alpha} (\frac{n}{k})^{1-\alpha} \geq C_\alpha n^{1-\alpha} \ln(n+1),$$

and similarly,

$$V_{\Lambda_1}(f_n) \geq C(\ln(n+1))^2.$$

Thus, if

$$f(x,y) = \sum_{k=1}^{\infty} a_k f_{n_k}(2^k x - 1, 2^k y - 1),$$

where the sequence of coefficients $\{a_k\}$ and the increasing sequence of natural numbers $\{n_k\}$ are appropriately chosen, we will have

$$f \in \Lambda_\alpha^* BV \setminus \Lambda_\alpha BV.$$

□

The general problem remains open.

4. Convergence of double Fourier series

Theorem 2 follows immediately from Theorem C and the following lemma.

Lemma 3. *Let* $\Lambda = \{\frac{n}{\ln(n+1)}\}_{n=1}^{\infty}, A = [a,b] \times [c,d]$. *Then* $\Lambda^* BV(A) \subset HBV(A)$.

Proof. Suppose $f \in \Lambda^* BV(A)$. Obviously $f(a,\cdot)$ and $f(\cdot,c)$ are in HBV. Let $\{I_i\}_{i=1}^N$ and $\{J_j\}_{j=1}^M$ be systems of nonoverlapping intervals in $[a,b]$ and $[c,d]$ respectively, and let $\Delta_{i,j} = |f(I_i \times J_j)|$. We enumerate the pairs $(i,j), i \in [1,N]$ and $j \in [1,M]$, as follows: assign 1 to $(1,1)$,and 2 and 3 to $(1,2)$ and $(2,1)$. Next we enumerate the (i,j) such that $i \cdot j = 3$ in any order, and so on. Let $\mu(i,j)$ denote the index corresponding to (i,j). For a given n, the number of (i,j) with $\mu(i,j) \geq 1$ and $i \cdot j \leq n$ is not greater than

$$\sum_{i=1}^{n} \frac{n}{i} \leq n \ln(n+1),$$

implying that, for these pairs, $\mu(i,j) \leq n\ln(n+1)$, and so

$$\lambda_{\mu(i,j)} \leq \frac{n\ln(n+1)}{\ln(n\ln(n+1)+1)} \leq 2n.$$

Thus, if $i \cdot j = n$, we have

$$\frac{\Delta_{i,j}}{i \cdot j} \leq \frac{2\Delta_{i,j}}{\lambda_{\mu(i,j)}}$$

for all (i,j). Thus

$$\sum_{i,j=1}^{i=N\ j=M} \frac{\Delta_{i,j}}{i \cdot j} \leq 2 \sum_{i,j=1}^{i=N\ j=M} \frac{\Delta_{i,j}}{\lambda_{\mu(i,j)}} \leq V_{\Lambda^*}(f),$$

which establishes Lemma 3. □

Remark 3. It is easily seen that $\Lambda^* BV(A)$ is a Banach space with $\|f\|_{\Lambda^*}$, as given in Definition 3(iii), as norm.

We turn now to the proof of Theorem 3.

Proof. Let N be a positive integer and $M = [N/2]$. For positive integers m and n such that $m \cdot n \leq M$, let

$$A_{mn} = \left[\frac{\pi(m-1)}{N+\frac{1}{2}}, \frac{\pi m}{N+\frac{1}{2}}\right) \times \left[\frac{\pi(n-1)}{N+\frac{1}{2}}, \frac{\pi n}{N+\frac{1}{2}}\right)$$

and set

$$g_N(x,y) = \sum_{m\cdot n \leq M} (-1)^{m+n} \chi_{A_{mn}}(x,y).$$

If A is a closed rectangle with sides parallel to the axes, then $g_n(A) \neq 0$ only when $A \cap A_{mn}$ contains exactly one vertex of A. Since the number of A_{mn} is not greater than $M\ln(M+1)$, we have

$$V_{\Lambda^*}(g_n) \leq \sum_{r=1}^{4M\ln(M+1)} \frac{\ln(r+1)}{r\xi_r} = o(\ln^2(M+1)),$$

and so

$$\|g_N\|_{\Lambda^*} = o(\ln^2(M+1)) = \eta_N \ln^2(M+1),$$

where $\eta_N = o(1)$ as $N \to \infty$, and, if

$$h_N = \frac{g_N}{\eta_N \ln^2(M+1)},$$

then $\{\|h_N\|_{\Lambda^*}\}$ is bounded.

If we now consider the square partial sums of the Fourier series of h_n at $(0,0)$, we have

$$\begin{aligned} \pi^2 S_{NN}[h_N;(0,0)] &= \frac{1}{\eta_N \ln^2(M+1)} \sum_{m\cdot n\le M} (-1)^{m+n} \iint_{A_{mn}} D_N(s)D_N(t)dsdt \\ &\ge \frac{4}{\eta_N \ln^2(M+1)\pi^2} \sum_{m\cdot n\le M} \frac{1}{m\cdot n} \ge C\frac{1}{\eta_N}\frac{\ln^2(M+1)}{\ln^2(M+1)} \to \infty \end{aligned}$$

as $N \to \infty, C$ being an absolute constant. Applying the Banach-Steinhaus theorem, we see that there must be an $f \in \Lambda^* BV$ such that $\{S_{NN}[f;(0,0)]\}$ diverges unboundedly. □

References

1. M. Avdispahić, Concepts of generalized bounded variation and the theory of Fourier series. Internat. J. Math. Math. Sci. **9** (1986), no. 2, 223–244. MR88c:42001
2. L. A. D'Antonio, D. Waterman, A summability method for Fourier series of functions of generalized bounded variation. Analysis **17** (1997), no. 2-3, 287–299. MR99h:42008
3. M. I. Dyachenko, Waterman classes and spherical partial sums of double Fourier series. Anal. Math. **21** (1995), no. 1, 3–21. MR97m:42009
4. C. Goffman, D. Waterman, The localization principle for double Fourier series. Studia Math. **69** (1980/81), no. 1, 41–57. MR82d:42010
5. A. A. Saakyan, On the convergence of double Fourier series of functions of bounded harmonic variation. (Russian) Izv. Akad. Nauk Armyan. SSR Ser. Mat. **21** (1986), no. 6, 517–529; English transl., Soviet J. Contemp. Math. Anal. **21** (1986), no. 6, 1–13. MR88j:42017
6. A. I. Sablin, Λ-variation and Fourier series. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1987, no. 10, 66–68; English transl., Soviet Math. (Iz. VUZ) 31 (1987), no. 10, 87–90. MR89c:42008
7. D. Waterman, On convergence of Fourier series of functions of generalized bounded variation. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity. II. Studia Math. **44** (1972), 107–117. MR46:9623
8. ———, On the summability of Fourier series of functions of Λ-bounded variation. Studia Math. **54** (1975/76), no. 1, 87–95 MR53:6212
9. ———, On Λ-bounded variation. Studia Math. **57** (1976), no. 1, 33–45. MR54:5408
10. ———, Fourier series of functions of Λ-bounded variation. Proc. Amer. Math. Soc. **74** (1979), no. 1, 119–123. MR80j:42010
11. ———, On some high-indices theorems. II. J. London Math. Soc. (2) **59** (1999), no. 3, 978–986. MR2000k:40006

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